Analogues to a Julia Boundary Away from Analyticity

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Dedicated to Prof. Ernst Bayer on the occasion of his 60th birthday

An analyticity-destroying term is introduced into the complex logistic map. Two types of self-similar basin boundaries are found, one a "coastline-type" direct analogue to Julia's, the other a locally smooth boundary that nevertheless is more complicated than the well-known "striated" fractal basin boundaries of Grebogi et al.

Complex boundaries in complex holomorphic maps are a well-studied field to date [1–5]. Recently it was proposed [6] that Smale's global analytic theory of "nontrivial basic sets" in differentiable dynamical systems [7] suffices to produce analogous boundaries in invertible 3-D maps and 4-D flows (of the hyperchaotic type introduced in [8]).

A straightforward first step to check this hypothesis numerically is to stay within the class of non-invertible dynamical systems, that is, 2-D maps, and sacrifice only the analyticity property.

The complex logistic map can in the simplest convention be written in the form

$$z_{n+1} = z_n^2 + c\,, \tag{1}$$

with $n \in \mathbb{N}$ and $z_n, c \in \mathbb{C}$. We introduce a real perturbing term to destroy analyticity, and obtain a two-dimensional real map

$$\begin{aligned} x_{n+1} &= x_n^2 - y_n^2 + a\,x_n + c_1\,, \\ y_{n+1} &= 2\,x_n y_n + c_2 \end{aligned} \tag{2}$$

with $x_n, y_n, a, c_1, c_2 \in \mathbb{R}$. Note that for $a = 0$ (2) is equivalent to (1).

In Fig. 1, a series of pictures is presented in which the parameter a varies from +0.00112, via zero, to −1.570875, with c_1 and c_2 held fixed ($c_1 = 0.32$, $c_2 = 0.043$). The pictures are plotted in the convention introduced by Mandelbrot [1], that is, points that diverge are shown in one color (say black), all others in the complementary one.

The first picture, Fig. 1 a, represents a disconnected "Fatou dust" [1] like structure. In the second (Fig. 1 b) the boundary of the basins is no longer disconnected. The third one, Fig. 1 c, is somewhat analogous (only the plotting convention is changed since the plotter favors the black points): It represents the well-known analytic case (cf. Map 18, p. 51, of [5]). The next picture (Fig. 1 d) shows a progressive "curling up" of the "white" subbasins. Shortly thereafter, in Fig. 1 e, all dividing strands have disappeared, leaving a single connected basin of one stable fixed point. This single basin then in the next few pictures (Figs. 1 f–h) sheds the most prominent feature of its boundary, the "hooks". In Fig. 1 i, they have disappeared altogether. At the same time the overall shape of the basin becomes pointed to the right, in order to eventually assume a more and more "triangular" circumference (Figs. 1 j, k). At $a = -1.30458\ldots$, the single fixed point becomes unstable via a period doubling bifurcation yet the basin remains connected (witness Figures 1 l, m). Shortly thereafter, the period of the orbit doubles again (at $a = -1.57020\ldots$). Eventually (apparently after a collision with an unstable cycle of period four) the period doubling sequence is stopped and all finite attractors disappear (Figures 1 o, p).

Two pictures of the present series were subjected to increasingly larger magnifications. The first series of blow-ups (Fig. 2) starts out from a magnification of part of Fig. 1 f, focusing on a side "hooklet" of one of the four major hooks visible there. The linear magnification factor ranges from 2,500 to $2.5 \cdot 10^{12}$. Obviously, each hook is covered by little hooklets that in turn are covered by even smaller ones, and so

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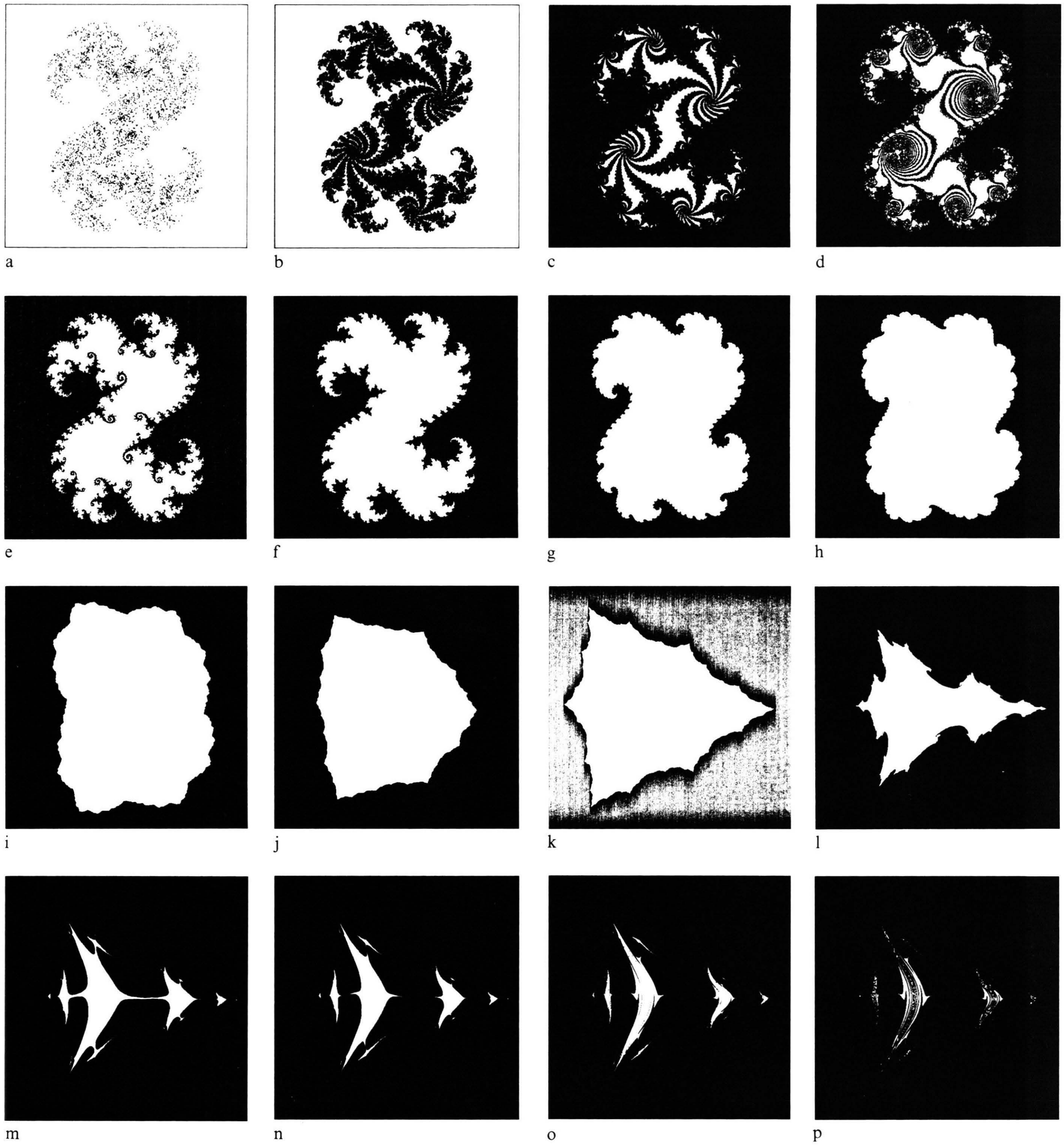


Fig. 1. Basin structure of (2), parametrized by different values of a (see Table 1); $c_1 = 0.32$ and $c_2 = 0.043$ were kept constant throughout. All calculations performed at 16 digits. Applied criterion for divergence: $x^2 + y^2$ exceeding 100. Maximum number of iteration steps per point: 1,000 to 25,000 (depending on whether or not picture stabilization had been obtained; the latter took longer near bifurcations, at a, d, o, and p). Inverse plot (the diverging solutions white) was used in a and b.

Table 1. Parameters and windows chosen in Figures 1–4.

Fig.	a	x_{min}	x_{max}	y_{min}	y_{max}
1a	0.00112	−1.25	1.25	−1.25	1.25
1b	0.0011	−1.25	1.25	−1.25	1.25
1c	0	−1.25	1.25	−1.25	1.25
1d	−0.0052	−1.25	1.25	−1.25	1.25
1e	−0.006	−1.25	1.25	−1.25	1.25
1f	−0.01	−1.25	1.25	−1.25	1.25
1g	−0.05	−1.25	1.25	−1.25	1.25
1h	−0.1	−1.25	1.25	−1.25	1.25
1i	−0.2	−1.25	1.25	−1.25	1.25
1j	−0.5	−1.25	1.75	−1.5	1.5
1k	−1	−1	2	−1.5	1.5
1l	−1.5	−1.5	2.5	−2	2
1m	−1.57	−1.5	2.5	−2	2
1n	−1.5705	−1.5	2.5	−2	2
1o	−1.57085	−1.5	2.5	−2	2
1p	−1.570875	−1.5	2.5	−2	2
2a	−0.01	−0.5548	−0.5558	0.2751	0.2761
2b	−0.01	−0.5540839	−0.5540849	0.2767458	0.2767468
2c	−0.01	−0.55408424	−0.55408425	0.27674523	0.27674524
2d	−0.01	−0.5540842367232	−0.5540842367242	0.2767452331318	0.2767452331328
3a	−1.57	1.4448	1.4458	0.3394	0.3404
3b	−1.57	1.445726	1.445727	0.3397765	0.3397775
3c	−1.57	1.44572652	1.44572653	0.339777176	0.339777186
3d	−1.57	1.4457265284344	1.4457265284354	0.3397771775958	0.3397771775968
4a	−0.0054506151	−1.25	1.25	−1.25	1.25
4b	−0.0055	−1.25	1.25	−1.25	1.25
4c	−0.00593	−1.25	1.25	−1.25	1.25
4d	−0.005937	−1.25	1.25	−1.25	1.25

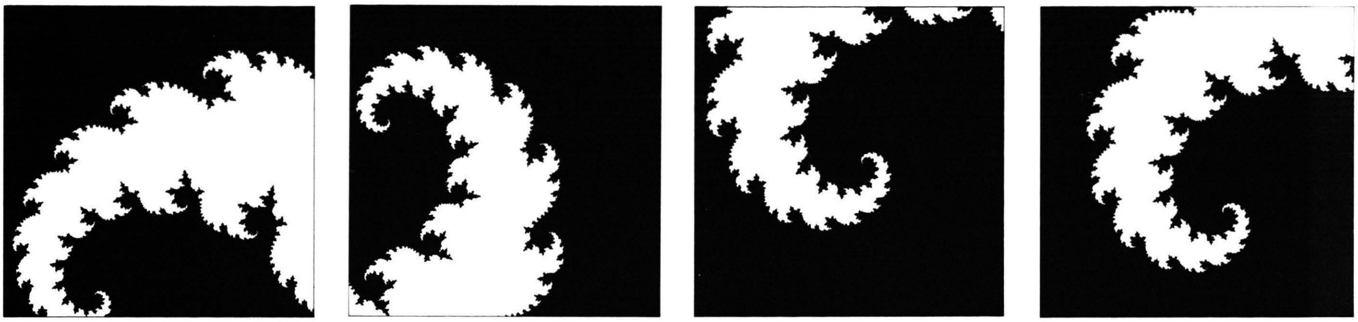
on. The same situation was found at other, neighboring pictures of Figure 1.

The second series of blow-ups (Fig. 3) visualizes the case of the "wild goose" of Figure 1m. Here subsequent magnifications of a side-protrusion from the top of the "middle triangle" of that Figure are shown. (Note that not always the "same" next-following protrusion was chosen.) Again, through all enlargements (up to $4 \cdot 10^{12}$), the qualitative shape remains the same. As before, neighboring pictures to Fig. 1m again showed the same qualitative behavior.

Thirdly and finally, one particular transition region, located within the sequence of pictures already shown in Fig. 1, appeared worth looking at a higher resolution of the parameter a. The pictures in Fig. 4 all lie within the "gap" between Fig. 1d and 1e. Figure 4a is similar to Figure 1d. It was chosen because at this particular value of a, the fixed point located in the major right-hand "hole", at X_a ($x = 0.420940896545\ldots$) and Y_a ($y = 0.271948441865\ldots$), becomes stable by way of an inverse Hopf bifurcation. Unexpectedly, at almost the same value of a (up to 3 digits), the attracting orbit of period 11 becomes locally unstable, that is, undergoes a Hopf bifurcation and gives way to a stable cycle of period 165. Then, presumably through a "collision" between the growing unstable torus around the fixed point and the stable cycle, all attracting periodic orbits disappear. In Fig. 4b one sees that the basin of the fixed point grows bigger while the other become smaller. Figs. 4c, d show further steps in the "break-up" of the bridges in between the different basins. Figure 1e follows 4d.

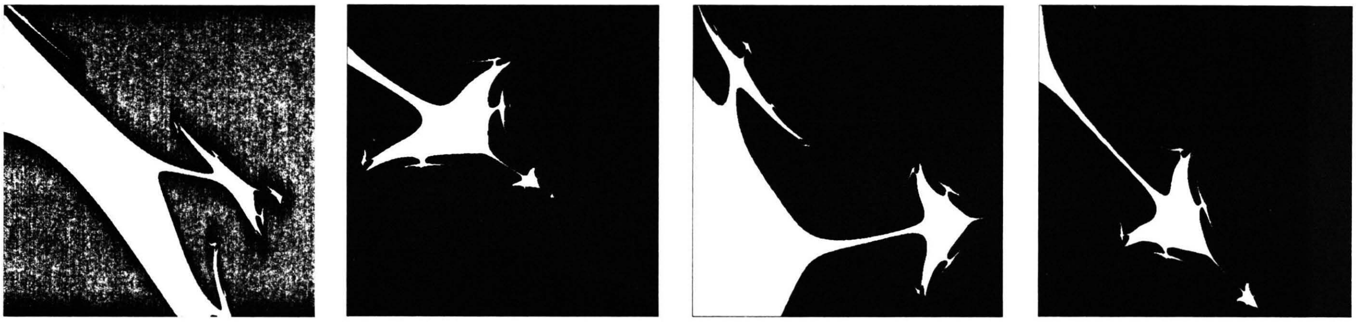
To conclude, two new features were presented – not counting the open questions just mentioned. The first is the "self-similar hook" structure of Figure 2. Like the analogous Julia boundary (see Fig. 36 of [5]), this structure is a fractal of the coastline type (cf. [1]).

The second new feature – found unexpectedly – is the "wild goose" of Figure 3. It too represents a self-similar structure of a basin boundary that is fractal "in two directions" [6, 10] in a sense. It on the other hand nevertheless contains an abundance of smooth (curvilinear) segments. It reminds one of



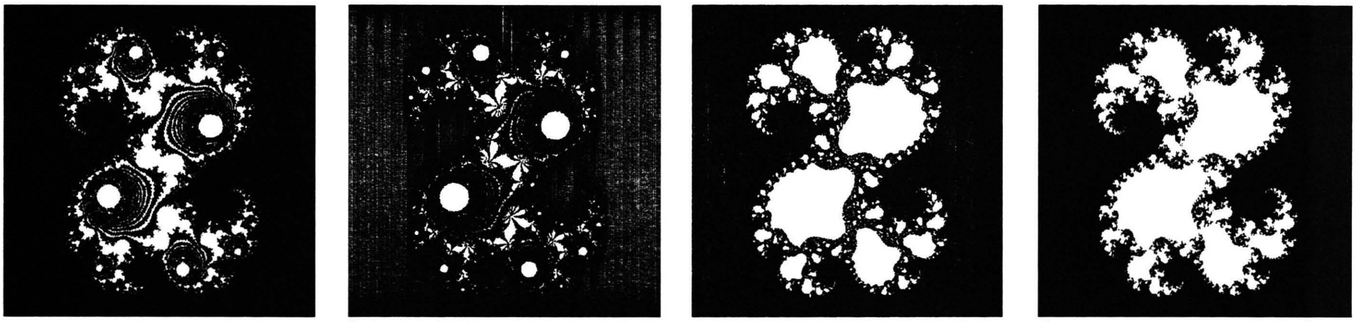
a b c d

Fig. 2. Blow-ups of Figure 1 f. See Table 1 for the windows used. Relative magnifications: (a): $2.5 \cdot 10^3$; (b): $2.5 \cdot 10^6$; (c): $2.5 \cdot 10^8$; and (d): $2.5 \cdot 10^{12}$.



a b c d

Fig. 3. Blow-ups of Fig. 1 m (see Table 1). Relative magnifications: (a): $4 \cdot 10^3$; (b): $4 \cdot 10^6$; (c): $4 \cdot 10^8$; and (d): $4 \cdot 10^{12}$.



a b c d

Fig. 4. Transition towards a single-basin structure. A finer-resolution window was taken out from Fig. 1 (in between d and e; see Table 1).

the "Alexandrov cross" type fractal boundary described in [6]. However, in that case the boundary contained infinitely many branch points (that is, was "web like" in character rather than "line like" as in the present case). A similar web like pattern occurs in some analytic maps (see the middle pictures on p. 134 of [5]). As the latter web breaks up into a single line, however (see the neighboring pictures on p. 134 of [5]), a coast-line type fractal is known to be formed. It therefore appears possible that the present type of boundary structure has no analogue in the analytic case.

Metzler, who studied coupled real logistic maps numerically [11, 12], already saw a structure that possibly is comparable: a self-similar pattern in parameter space in which identical-looking objects – each with a smooth surface structure – occurred inside of each other on all scales in the manner of a Chinese doll (see Fig. 8 of [12]). The present object, however, apart from its belonging to phase space rather than parameter space, is connected.

The type of self-similarity seen in Fig. 3 differs from that seen in Fig. 2 (the familiar one) in much the same way as a Cantor set differs from its complement. Both are fractal objects, and both are maximally self-similar in the dimension in question. Yet one contains featureless pieces – smooth segments – and the other does not. The boundary structure in Fig. 3 indeed resembles more a Cantor function ("devil's staircase"; cf. [13]) than an ordinary coastline. On the other hand its fractal nature nevertheless contrasts markedly with the well known "fractal basin boundaries" described by Grebogi et al. [14, 15] and Mira [16] which possess a "striated" (locally stripe-like) character of the type also seen in Figure 1 p.

It therefore appears that at least three "major" types of fractal basin boundaries (coastline, wild goose, striated) have to be distinguished in 2-D endomorphisms.

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